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On Diffuse Reflection at the Boundary for the Boltzmann Equation and Related Equations

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The paper considers diffuse reflection at the boundary with nonconstant boundary temperature and unbounded velocities. The solutions obtained are proved to conserve mass at the boundary. After a preliminary study of the collisionless case, the main results obtained are existence for the Boltzmann equation in a "DiPerna–Lions framework" with the above boundary conditions in a bounded measure sense, and existence together with uniqueness for the BGK equation with Maxwellian diffusion on the boundary in an L^∞ framework.

KEY WORDS: Boltzmann equation; BGK equation; diffuse reflection; boundary value problem.

1. INTRODUCTION

This paper considers initial boundary value problems for the Boltzmann equation and the BGK equation, when the behavior at the boundary is governed by diffuse reflection, sufficiently similar to Maxwellian diffuse reflection for formal conservation of mass at the boundary.

For the corresponding collisionless problem a great deal of information for large data is available; see, e.g., refs. 11 and 6 for details and references. As for the nonlinear Boltzmann equation, a first treatment of the present diffuse boundary behavior in a DiPerna–Lions setting was given by Hamdache.[7] This was later extended[1] in several directions including the case of general diffuse reflection with varying boundary temperature under a restriction to bounded velocities. For an extensive discussion of the background and references for the problem we refer to ref. 7.

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In the present paper we introduce a class of boundary operators (in generality intermediate between those in ref. 1 and pure Maxwellian diffuse reflection) for which the restriction to bounded velocities can be removed also when the boundary temperature is allowed to vary. In a preliminary discussion, a collection of relevant (old and new) results on the problem with known gain term and collision frequency is presented. This is followed by a study of the Boltzmann case. The paper ends with an analysis of the situation for the BGK model.

The problems considered in this paper are all of the type

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f = Qf, \qquad t \in (0, T), \quad x \in \Omega, \quad \xi \in \mathbb{R}^3 \tag{1.1}$$

$$f(t, x, \xi) = Kf = \int_{\xi' \cdot n(x) < 0} K(\xi' \to \xi, t, x)\, f t, x, \xi')\, d\xi' \tag{1.2}$$

$$t \in (0, T), \qquad x \in \partial\Omega, \qquad \xi \cdot n(x) > 0$$

$$f(0, x, \xi) = f_0(x, \xi), \qquad x \in \Omega, \quad \xi \in \mathbb{R}^3 \tag{1.3}$$

Here $\Omega \subset \mathbb{R}^d$, $d \leqslant 3$, is open with a smooth boundary $\partial\Omega$ (of Lyapunov type is sufficient). The unit inward normal to $\partial\Omega$ is $n(x)$, and the measure of Ω and $\partial\Omega$ are finite.

A classic example of (2) is Maxwellian diffuse reflection

$$K(\xi' \to \xi; t, x) = |\xi' \cdot n(x)|\, M(t, x, \xi) \tag{1.4}$$

where

$$M(t, x, \xi) = (2\pi)^{-2}\, \Theta^2 \exp(-0.5\Theta\, |\xi|^2) \tag{1.5}$$

with prescribed inverse temperature $\Theta(t, x)$ such that

$$0 < c_1 < \Theta(t, x) < c_2 < \infty$$

In the preliminary Section 2, a reformulation of the problem (1.1)–(1.3) is introduced together with definitions and notations. Section 3 collects the properties needed for the boundary traces. Section 4 introduces the type of diffuse reflection operators used, here referred to as regular diffuse operators, and presents some of their properties. In Section 5 variants of the problem with known gain term and collision frequency are discussed. This is then used for the proof of existence of solutions to the regular diffuse initial boundary value problem in the case of the Boltzmann equation in Section 6, and of existence and uniqueness for the corresponding BGK problem in Section 7. In particular, the latter section contains the observations needed to confirm the statement at the end of ref. 8 in the case of the boundary condition (1.4).

2. PRELIMINARIES

For the convenience of the reader we collect notations of a standard nature connected to our problem in the beginning of this section. We have

$$\mathfrak{r} = (t, x, \xi)$$

is an element in the space

$$\mathscr{D} = (0, T) \times \Omega \times \mathbb{R}^3$$

Lower-dimensional regions of interest are

$$\Gamma^{\mp} = \{\mathfrak{r} \in \bar{\mathscr{D}};\, x \in \partial\Omega,\, \xi \cdot n(x) \lesseqgtr 0\}$$

$$\Gamma^{s} = \{\mathfrak{r} \in \bar{\mathscr{D}};\, t = s\}$$

with characteristic functions

$$\chi^{\pm} = \mathbf{1}_{\Gamma^{\pm}}, \qquad \chi^{s} = \mathbf{1}_{\Gamma^{s}}$$

The boundaries of $\mathscr{D}$ are

$$\partial\mathscr{D}^{+} = \Gamma^{+} \cup \Gamma^{0}, \qquad \partial\mathscr{D}^{-} = \Gamma^{-} \cup \Gamma^{T}$$

Define the backward and forward stay times as

$$t^{+} = t^{+}(\mathfrak{r}) = \inf\{s > 0;\, x - s\xi \in \partial\Omega\}$$

$$t^{-} = t^{-}(\mathfrak{r}) = \inf\{s > 0;\, x + s\xi \in \partial\Omega\}$$

with the related quantities

$$s^{+}(\mathfrak{r}) = \min\{t,\, t^{+}(\mathfrak{r})\}$$

$$s^{-}(\mathfrak{r}) = \min\{T - t,\, t^{-}(\mathfrak{r})\}$$

Reparametrizations of $\mathscr{D}$ employing these quantities are

$$\mathscr{D}^{\pm} = \{(s, \mathfrak{r});\, \mathfrak{r} \in \partial\mathscr{D}^{\pm},\, s \in (-s^{+}(\mathfrak{r}), s^{-}(\mathfrak{r}))\}$$

Let $d\sigma$ denote the usual measure on $\partial\Omega$, and

$$d\sigma_{\mathfrak{r}}^{\pm} = \begin{cases} |\xi \cdot n(x)|\; dt\; d\sigma\; d\xi, & \mathfrak{r} \in \Gamma^{\pm} \\ dx\; dt, & \mathfrak{r} \in \Gamma^{s} \quad \text{for} \quad s = 0 \quad \text{resp. } T \end{cases}$$

The following scalar products are used:

$$\langle f, g\rangle = \int_{\mathscr{D}} fg\, d\mathfrak{r}$$

$$\langle f, g\rangle_{\pm} = \int_{\partial\mathscr{D}^{\pm}} fg\, d\sigma_{\mathfrak{r}}^{\pm}$$

$$\langle f, g\rangle_{t} = \int_{\Omega\times\mathbb{R}^3} f(t, x, \xi)\, g(t, x, \xi)\, dx\, d\xi$$

$$(f, g) = \int_{\mathbb{R}^3} fg\, d\xi$$

$$(f, g)_{\pm} = \int_{\mathbb{R}^3} fg\chi^{\pm}\, |\xi\cdot n(x)|\, d\xi$$

Some related spaces are

$$L^{p\pm} = L^p(\partial\mathscr{D}^{\pm}, d\sigma^{\pm})$$

$$L^p(\Gamma^{\pm}) = L^p(\Gamma^{\pm}, d\sigma^{\pm})$$

$$W^p = \{f\in L^p(\mathscr{D});\ \wedge f\in L^p(\mathscr{D})\} \qquad \text{with} \quad \wedge = \frac{\partial}{\partial t} + \xi\cdot\nabla_x$$

We also use the space $\mathscr{M}^{\pm}$ of σ-finite measures defined on the σ-algebra $\mathscr{B}^{\pm}$ of Borel sets from $\Gamma^{\pm}$. The notation $\langle\varphi, \mu^{\pm}\rangle_{\pm}$ is used for the corresponding integrals

$$\langle\varphi, \mu^{\pm}\rangle_{\pm} = \int_{\Gamma^{\pm}} \varphi\, d\mu^{\pm}$$

with $\varphi\in C_0(\Gamma^{\pm})$, $\mu^{\pm}\in\mathscr{M}^{\pm}$.

The mappings

$$R^{\pm}: \mathscr{D}^{\pm}\to\mathscr{D}$$

with

$$R^{\pm}(s, \mathfrak{r}) = R(s, \mathfrak{r}) = (t+s, x+s\xi, \xi)$$

define characteristic coordinates $(s, \mathfrak{r})$ for functions f defined on $\mathscr{D}$,

$$f^{\#}(s, \mathfrak{r}) = f^{\#}(s) = f(R(s, \mathfrak{r}))$$

Also, for functions φ defined on $\partial\mathscr{D}^{\pm}$

$$(R^{\pm}\varphi)^{\#}(s, \mathbf{r}) = \varphi(\mathbf{r})$$

i.e., an $\mathbb{R}^{\pm}$ extension of φ along characteristics. For $f \in L^1(\mathscr{D})$

$$\int_{\mathscr{D}} f(\mathbf{r})\, d\mathbf{r} = \int_{\partial\mathscr{D}^+} \left[\int_0^{s^-(\mathbf{r})} f^{\#}(s, \mathbf{r})\, ds\right] d\sigma_{\mathbf{r}}^+$$

$$\int_{\mathscr{D}} f(\mathbf{r})\, d\mathbf{r} = \int_{\partial\mathscr{D}^-} \left[\int_{-s^+(\mathbf{r})}^{0} f^{\#}(s, \mathbf{r})\, ds\right] d\sigma_{\mathbf{r}}^-$$

The trace studies will use

$$\hat{W}^p = \{f \in W^p; \gamma^{\pm} f \in L^{p^{\pm}}\}, \qquad 1 \leqslant p \leqslant \infty$$

Here the trace operators $\gamma^{\pm}$ are defined by

$$\gamma^{\pm} f = \lim_{s \to \mp s^{\pm}(\mathbf{r})} f^{\#}(s, \mathbf{r}) \tag{2.1}$$

The following sort of Green's formula holds[10]:

$$\langle f, g \rangle_- - \langle f, g \rangle_+ = \langle f, \Lambda g \rangle + \langle g, \Lambda f \rangle \tag{2.2}$$

for $f \in \hat{W}^p$, $g \in \hat{W}^{p'}$, $1 \leqslant p < \infty$.

The collision operator is Q with gain and loss term Q^+, resp. Q^-,

$$Qf = Q^+ f - Q^- f, \qquad Q^{-f} = f\nu(f)$$

Sometimes Q^+ and ν are given functions independent of f.

Given $\mathbf{r} \in \bar{\mathscr{D}}$, τ, $s \in [-s^+(\mathbf{r}), s^-(\mathbf{r})]$, (1.1) can formally be rewritten as

$$f^{\#}(s, \mathbf{r}) = f^{\#}(\tau, \mathbf{r})\, \Pi(\tau, s, \mathbf{r}) + \int_\tau^s (Q^+ f)^{\#}(z, \mathbf{r})\, \Pi(z, s, \mathbf{r})\, dz \tag{2.3}$$

with

$$\Pi(\nu; \tau, s, \mathbf{r}) = \Pi(\tau, s, \mathbf{r}) = \exp\left[-\int_\tau^s \nu^{\#}(z, \mathbf{r})\, dz\right]$$

Also set

$$\Pi^+(\nu; \tau, s, \mathbf{r}) = \Pi^+(\tau, s, \mathbf{r}) = \exp\left[\int_\tau^s \nu^{\#}(z, \mathbf{r})\, dz\right]$$

The following conversion of (1.1)–(1.3) will frequently be used. Set

$$Z = \{f_0, g, G, v\}$$

for f_0: $\Omega \times \mathbb{R}^3 \to \mathbb{R}$, g: $\Gamma^+ \to \mathbb{R}$, G: $\mathscr{D} \to \mathbb{R}$, v: $\mathscr{D} \to \mathbb{R}^+$. Using (2.3) with $\mathfrak{r} \in \partial\mathscr{D}^+$, $\tau = 0$, $s \in [0, s^-(\mathfrak{r})]$, we find that (1.1) formally gives

$$f = V(Z)$$

and

$$V^{\#}(Z; s, \mathfrak{r}) = \Pi(0, s, \mathfrak{r})[f_0(\mathfrak{r})\, \chi^0(\mathfrak{r}) + g(\mathfrak{r})\, \chi^+(\mathfrak{r})] + \int_0^s G^{\#}(z, \mathfrak{r})\, \Pi(z, s, \mathfrak{r})\, dz \tag{2.4}$$

where

$$Z = (f_0, f^+, Q^+f, v(f)), \qquad f^+ = \chi^+\gamma^+ f$$

Using (2.3) with $\mathfrak{r}$ replaced by $\mathfrak{r}' = (t, x, \xi')$, $s = 0$, $\tau = -s^+(\mathfrak{r}')$, and $\mathfrak{r}' \in \Gamma^-$ formally gives

$$f^+ = \mathscr{K}(Z) \tag{2.5}$$

Here

$$f^+ = \chi^+\gamma^+ f$$

and

$$\begin{aligned}\mathscr{K}(Z, \mathfrak{r}) = \int_{\xi' \cdot n(x) < 0} K(\xi' \to \xi; t, x) \Big\{ &[f_0^{\#}(-s^{+'}, \mathfrak{r}')\, \chi^0(x - s^{+'}\xi') \\ &+ f^{+\#}(-s^{+'}, \mathfrak{r}')\, \chi^+(x - s^{+'}\xi')]\, \Pi(-s^{+'}, 0, \mathfrak{r}') \\ &+ \int_{-s^{+'}}^0 G^{\#}(y, \mathfrak{r}')\, \Pi(y, 0, \mathfrak{r}')\, dy \Big\}\, d\xi' \end{aligned} \tag{2.6}$$

$$(t, x, \xi) = \mathfrak{r} \in \Gamma^+, \qquad \mathfrak{r}' = (t, x, \xi'), \qquad s^{+'} = s^+(\mathfrak{r}')$$

We thus have the system

$$f = V(f_0, f^+, Q^+f, v(f)) \tag{2.7}$$

$$f^+ = \mathscr{K}(f_0, f^+, Q^+f, v(f)) \tag{2.8}$$

for f: $\mathscr{D} \to \mathbb{R}$, f^+: $\Gamma^+ \to \mathbb{R}$.

Conversely, if (f, f^+) solves (2.7)–(2.8), then formally $\chi^+\gamma^+ f = f^+$ and f solves (1.1)–(1.3). Let

$$\psi \in L^\infty(\mathscr{D}), \qquad \eta \in L^{\infty -}, \qquad \varphi \in L^{\infty +} \tag{2.9}$$

with compact supports. Green's formula (2.2) gives

$$\langle \psi, V(Z)\rangle = \langle \chi^0 f_0 + \chi^+ f^+, \gamma^+ f^+, \gamma^+ V^*(v, \psi)\rangle_+ + \langle Q^+, V^*(v, \psi)\rangle \tag{2.10}$$

$$\langle \eta, V(Z)\rangle_- = \langle \chi^0 f_0 + \chi^+ f^+, \gamma^+ V^*_-(v, \eta)\rangle_+ + \langle Q^+, V^*_-(v, \eta)\rangle \tag{2.11}$$

$$\langle \varphi, \mathscr{K}(Z)\rangle_+ = \langle \chi^0 f_0 + \chi^+ f^+, \gamma^+ \mathscr{K}^*(v, \varphi)\rangle_+ + \langle Q^+, \mathscr{K}^*(v, \varphi)\rangle \tag{2.12}$$

This is so, because

$$V^*(v, \psi)^\# (s, \mathrm{r}) = \int_s^{s^-(\mathrm{r})} \psi^\#(\tau, \mathrm{r})\, \Pi(s, \tau, \mathrm{r})\, d\tau \tag{2.13}$$

solves

$$\wedge g - vg = -\psi, \qquad \gamma^- g = 0$$

and

$$V^*_-(v, \eta)^\# (s, \mathrm{r}) = \Pi(s, s^-(\mathrm{r}), \mathrm{r})(R^-\eta)^\# (s, \mathrm{r}) \tag{2.14}$$

solves

$$\wedge g - vg = 0, \qquad \gamma^- g = \eta$$

Finally, setting $\eta = \mathscr{K}^*\varphi$ in (2.11), we obtain (2.12) with

$$\mathscr{K}^*(v, \varphi)^\# (y, \mathrm{r}') = \Pi(y, 0, \mathrm{r}')(R^-(K^*\varphi))^\# (y, \mathrm{r}') \tag{2.15}$$

where

$$K^*\varphi(\mathrm{r}') = \int_{\xi \cdot n(x) > 0} \varphi(\mathrm{r})\, K(\xi' \to \xi; t, x)\, |\xi \cdot n(x)|/|\xi' \cdot n(x)|\, d\xi \tag{2.16}$$

Set $\varphi = 1$ in (2.12) and use (2.15) to get the following result.

Lemma 2.1. Assume that $v \in L^1_{\mathrm{loc}}(\mathscr{D})$, $v \geqslant 0$,

$$K^*\varphi \geqslant 0 \qquad \text{for} \quad \varphi \geqslant 0, \qquad \sup_{\Gamma^-}(K^*1) \leqslant \bar{K} \leqslant 1 \tag{2.17}$$

Then $\mathscr{K}$ is continuous from $L^1(\Omega \times \mathbb{R}^3) \times L^1(\Gamma^+) \times L^1(\mathscr{D})$ into $L^1(\Gamma^+)$, and

$$\|\mathscr{K}(Z)\|_{L^1(\Gamma^+)} \leqslant \bar{K}(\|f_0\|_{L^1(\Omega \times \mathbb{R}^3)} + \|f^+\|_{L^1(\Gamma^+)} + \|Q^+\|_{L^1(\mathscr{D})})$$

Let us also assume that

$$K^*: \quad C_0(\Gamma^+) \to C_b(\Gamma^-)$$

Then we can define $K\mu^- \in \mathcal{M}^+$ for measures $\mu^- \in \mathcal{M}^-$ through

$$\langle \varphi, K\mu^- \rangle_+ = \langle K^*\varphi, \mu^- \rangle_-$$

The boundary condition (1.2) for measures in our case becomes

$$\mu^+ = K\mu^- \tag{2.18}$$

By the Lebesgue decomposition theorem there are

$$\mu_s^\pm \in \mathcal{M}^\pm, \qquad N^\pm \subset \Gamma^\pm$$

with $\sigma^\pm(N^\pm) = 0$ and measurable function $f^\pm = d\mu^\pm/d\sigma^\pm$ such that

$$\mu^\pm(A) = \int_A f^\pm \, d\sigma^\pm + \mu_s^\pm(A)$$

$$\mu_s^\pm(\Gamma^\pm \backslash N^\pm) = 0$$

for all measurable sets $A \subset \Gamma^\pm$.

We shall in the Boltzmann equation case consider diffuse reflection operators for measures $\mu^\pm$ satisfying (2.18) together with

$$\frac{d\mu^\pm}{d\sigma^\pm} \geqslant \gamma^\pm f \tag{2.19}$$

Definition 2.1. f is a mild solution of (1.1)–(1.3) if

$$f \in L^1(\mathscr{D}), \qquad f \geqslant 0, \qquad (Q^\pm f)^\# \in L^1([0, s^-]) \tag{2.20}$$

$$f^\#(s, \mathrm{r}) = f^\#(\tau, \mathrm{r}) + \int_\tau^s Q^\#(z, \mathrm{r})\, dz, \qquad 0 \leqslant s < \tau \leqslant s^-(\mathrm{r}) \tag{2.21}$$

$$\chi^0 f = f_0$$

for a.e. $\mathrm{r} \in \partial\mathscr{D}^+$, and there are $\mu^\pm \in \mathcal{M}^\pm$ satisfying (2.18) and (2.19).

Definition 2.2. f is a solution in exponential multiplier form (or exponential solution for short) of (1.1)–(1.3) if

$$f \in L^1(\mathscr{D}), \qquad f \geqslant 0, \qquad \nu(f) \in L^1_{\mathrm{loc}}(\mathscr{D}) \tag{2.22}$$

$$f^\#(s, \mathrm{r}) = V^\#(f_0, \gamma^+ f, Q^+ f, \nu(f))(s, \mathrm{r}), \qquad 0 \leqslant s \leqslant s^-(\mathrm{r}) \tag{2.23}$$

for a.e. $\mathrm{r} \in \partial\mathscr{D}^+$, and if there exist $\mu^+ \in \mathcal{M}^\pm$ such that (2.18)–(2.19) hold.

Renormalized solutions are defined similarly. Equivalence relations similar to those in the Cauchy problem[4] hold. In particular, assume that

$$0 \leqslant f \in L^1(\mathscr{D}), \qquad \nu(f) \in L^1_{\text{loc}}(\mathscr{D})$$

Then f is a mild solution if and only if f is an exponential solution. With $\mu_c^{\pm} = f^{\pm}\, d\sigma^{\pm}$, it follows from (2.18) that

$$\mu_c^+ = (K\mu_s^-)_c + K\mu_c^-$$
$$\mu_s^+ = (K\mu_s^-)_s$$

Hence

$$f^+ \geqslant Kf^- \tag{2.24}$$

3. ON THE TRACES

It follows from (2.3) that any exponential solution of (1.1)–(1.3) has traces $\gamma^{\pm} f$, defined a.e. on $\partial D^{\pm}$ by (2.1). For "nice" solutions the trace operators are determined in the usual sense of the trace theory. Namely, let

$$\gamma^{\pm} f = f|_{\partial\mathscr{D}} \qquad \text{for} \quad f \in C(\bar{\mathscr{D}}) \tag{3.1}$$

Lemma 3.1.[10] There exist continuous extensions of (3.1)

$$s^{\mp}\gamma^{\pm}: \quad W^1 \to L^{1\pm}$$

such that

$$|\langle s^{\pm}, \gamma^{\mp} f\rangle_{\mp} - \langle f, 1\rangle| \leqslant T\, \|\Lambda f\|_{L^1(\mathscr{D})} \tag{3.2}$$

Proof. Set $g^{\pm}(s, \mathfrak{r}) = s \pm s^{\pm}(\mathfrak{r})$. Use (2.2) to obtain

$$\{s^+, \gamma^- f\rangle_- = \langle f, 1\rangle + \langle s + s^+, \Lambda f\rangle$$
$$\langle s^-, \gamma^+ f\rangle_+ = \langle f, 1\rangle + \langle s - s^-, \Lambda f\rangle$$

The lemma follows. ∎

Remark. The operators (3.1) have continuous extensions[10]

$$\gamma^{\pm}: \qquad \hat{W}^{1\mp} \to L^{1\pm}$$

and

$$\|\gamma^- f\|_{L^{1-}} = \|\gamma^+ f\|_{L^{1+}} + \langle \Lambda f, 1\rangle$$

ξ^2-traces are also of interest. The following result holds.

Lemma 3.2. With $\tilde{\Lambda} = \xi \nabla_x$, $\tilde{\chi}^{\pm} = 1_{\xi \cdot n(x) \gtrless 0}$, set

$$\tilde{W}^1 = \{ f \in W^1(\Omega \times \mathbb{R}^3);\ |\xi|^2 f,\ \xi \tilde{\Lambda} f \in L^1(\Omega \times \mathbb{R}^3) \}$$
$$\tilde{L}^{1\pm} = L^1(\partial\Omega \times \mathbb{R}^3, \tilde{\chi}^{\pm}\ |\xi \cdot n(x)|\ d\sigma\ d\xi)$$

The operators

$$\tilde{\gamma}^{\pm} f = \tilde{\chi}^{\pm} f|_{\partial\Omega \times \mathbb{R}^3} \qquad \text{for} \quad f \in C(\bar{\Omega} \times \mathbb{R}^3)$$

have continuous extensions

$$|\xi \cdot n(x)|\ \tilde{\gamma}^{\pm}: \quad \tilde{W}^1 \to \tilde{L}^{1\pm}$$

Proof. Take $f \in \tilde{W}^1$ and set $g = -u \cdot \xi$, where $u \in (C^1(\bar{\Omega}))^3$, $u(x) = n(x)$ for $x \in \partial\Omega$. By approximation Green's formula (2.2) implies

$$- \int_{\partial\Omega \times \mathbb{R}^3} f g \xi \cdot n(x)\ d\sigma\ d\xi = \langle f, \tilde{\Lambda} g \rangle_0 + \langle g, \tilde{\Lambda} f \rangle_0$$

and so the lemma follows. ∎

Remark. Similarly, consider a nonnegative function $f \in W^1(\mathscr{D})$ with $\xi^2 f \in L^1(\mathscr{D})$, $|\xi|\, f \in L^1(\Gamma^s)$, $s = 0, T$, and with

$$(\Lambda f, \psi) = 0 \qquad \text{for} \quad \psi = 1, \xi, |\xi|^2 \tag{3.3}$$

It follows by Green's formula (2.2) that for $g = -\xi \cdot u$ with u as in the proof of Lemma 3.2,

$$\begin{aligned} &\langle f, \chi^+\ |\xi \cdot n(x)| \rangle_+ + \langle f, \chi^-\ |\xi \cdot n(x)| \rangle_- \\ &\qquad \leqslant C[\langle f, |\xi|^2 \rangle + \langle f, |\xi| \rangle_T + \langle f, |\xi| \rangle_0] \end{aligned} \tag{3.4}$$

If (3.3) does not hold, then the term $\langle |\xi|, |\Lambda f| \rangle$ should be added to the right-hand side.

4. REGULAR REFLECTION OPERATORS AND *A PRIORI* ESTIMATES

In this section we study a class of boundary operators for which (1.2) holds with equality and conservation of mass flux. Restrictions like the ones below on K^* seem necessary to control various fluxes at the boundaries, when these are not isothermal. Still stronger restrictions are needed if more regularity is required at the boundary, as for the BGK study in the final section of the paper.

Let K^* be defined by (2.16). The assumptions (2.17) already control the sign of the gas density and prevent injection of mass from the boundary. Control of mass, energy, and entropy flows for the outgoing distributions is provided by the following conditions:

(K_0) (Sign control) $K^*\psi \geqslant 0$ for $\psi \geqslant 0$.

(K_1) (Mass condition) $K^*1 = 1$.

(K_2) (Spreading condition) There exists $K_2 > 0$ such that

$$K^* \, |\xi \cdot n(x)| \geqslant K_2$$

(K_3) (Energy condition) There exists $K_3 < \infty$ such that

$$K^* \, |\xi|^2 \leqslant K_3$$

(K_4) (Entropy condition) There exist $K_4 < \infty\}$ and $\alpha \in [0, 1)$ such that for every $f \in L^1(\Gamma^-)$ with $f \geqslant 0$,

$$\left\langle Kf, \ln \frac{Kf}{(f, 1)_+} \right\rangle_+ - \alpha \hat{H}^- \leqslant K_4(q_2^- + q)$$

Here

$$\hat{H}^- = \left\langle \chi^- f, \ln \frac{f}{(f, 1)_-} \right\rangle_-, \qquad q_j^\pm = \langle \chi^\pm f, |\xi|^j \rangle_\pm$$

$$q = \langle |\xi \cdot n|, \chi^+ f \rangle_+ + \langle |\xi \cdot n|, \chi^{-f} \rangle_-$$

(K_5) Spreading condition) There exists a decreasing function $\psi \in C((0, \infty), (0, 1))$ such that

$$K^* 1_{t^- > s} \geqslant \psi(s) \qquad \text{for} \quad 0 < s$$

Notice that for Maxwellian diffuse reflection

$$K^* \, |\xi \cdot n(x)| = C\Theta(t, x)^{-1/2}$$

$$K^* \, |\xi|^2 = C\Theta(t, x)^{-1}$$

$$\left\langle Kf, \ln \frac{Kf}{(f, 1)_+} \right\rangle_+ = \left(2 \ln \frac{\Theta}{2\pi} - C \right) \langle \chi^- f, 1 \rangle_-$$

with absolute constants C.

On the other hand, (K_2) excludes specular and reverse reflection operators. The condition (K_4) holds under (K_2) together with

$$\left\langle Kf, \ln \frac{Kf}{(f, 1)_+} \right\rangle_+ \leqslant K_4 \langle f, 1 + |\xi|^2 \rangle_-$$

The spreading condition (K_5) introduces some restriction on $\partial\Omega$, essentially excluding exotic domains for which the measure is zero of the part of the boundary available from some point x. It may be used for investigation of slowly decreasing functions f with $\langle|\xi|^2, f\rangle = \infty$ (see Theorem 5.2 below).

Set

$$m_j(t) = \langle f, |\xi|^j\rangle_t$$
$$H(t) = \langle f, \ln f\rangle_t$$
$$H^{\pm} = \langle \chi^{\pm} f, \ln f\rangle_{\pm}$$

In the following lemmas, some properties of solutions to (1.1)–(1.3) connected to (K_1)–(K_5) are presented.

Lemma 4.1. Assume that (K_0)–(K_3) hold. Let f be an exponential solution of (1.1)–(1.3) with equality in (2.19) and

$$\mu_s^- = 0, \qquad 0 \leqslant (1+|\xi|^2)\, Q^+ \in L^1(\mathscr{D})$$
$$0 \leqslant (1+|\xi|^2)\, f_0 \in L^1(\Omega\times\mathbb{R}^3), \qquad v \in L^1_{\mathrm{loc}}(\mathscr{D})$$

If $(Q(f), 1) = 0$ for f an exponential solution, then f satisfies

$$m_0(t) = m_0(0) \tag{4.1}$$

If $(Q(f), \psi) = 0$ for f an exponential solution and $\psi = 1,\ \xi,\ |\xi|^2$, then f satisfies

$$\sup_{t\leqslant T} m_2(t) + q_0^+ + q_0^- + q_2^+ + q_2^- \leqslant C_1(T) \tag{4.2}$$

with $C_1(T) > 0$ only depending on f_0 and K_2, K_3.

If $\langle Q^+(f), 1\rangle \leqslant c_1 + c_2\langle f, 1\rangle$, then

$$m_0(t) \leqslant C_2(T) \tag{4.3}$$

with $C_2(T) > 0$ only depending on f_0, c_1, and c_2.

If $\langle Q^+(f) + Q^-(f),\ 1+\xi^2\rangle \leqslant c_3 + c_4\langle f, 1+\xi^2\rangle$, then (4.2) holds with $C_1(T)$ only depending on f_0, K_2, K_3, c_3, and c_4.

Proof. The condition (K_1) implies

$$q_0^- = q_0^+$$

and so if $(Q, 1) = 0$, then by (2.2)

$$m_0(T) = m_0(0)$$

If $(Q, |\xi|^2)=0$, then analogously

$$m_2(T)+q_2^- - q_2^+ = m_2(0) \tag{4.4}$$

It follows from (K_2) that

$$\langle Kf, |\xi\cdot n|\rangle_+ = \langle \chi^- f, K^* |\xi\cdot n|\rangle_- \geqslant K_2 q_0^-$$

By (3.4) this implies

$$\langle \chi^- f, 1\rangle_- \leqslant C\left[\int_0^T m_2(t)\,dt + m_1(T)+m_1(0)\right] \tag{4.5}$$

if $(Q, \xi)=0$. Condition (K_3) gives

$$q_2^+ = \langle Kf, |\xi|^2\rangle_+ = \langle f, K^*\,|\xi|^2\rangle_- \leqslant K_3\langle \chi^- f, 1\rangle_- \tag{4.6}$$

Together (4.4)–(4.6) imply

$$m_2(T)\leqslant C\left[\int_0^T m_2(t)\,dt + m_1(T)+m_1(0)\right]+m_2(0)$$

Together with the inequality

$$m_1(T)\leqslant \varepsilon m_2(T)+C(\varepsilon)\,m_0(T)$$

this gives

$$m_2(T)\leqslant C(T)$$

by Gronwall's lemma, and (4.1) follows. The other cases are proved similarly. ∎

Lemma 4.2. Keep the conditions of the previous lemma and assume in addition (K_4), together with

$$(f_0, \ln f_0)\in L^1(\Omega) \tag{4.7}$$

$$(Q(f), \ln f)\leqslant 0, \qquad (Q(f), \psi)=0 \qquad \text{for} \quad \psi = 1, \xi, |\xi|^2 \tag{4.8}$$

Then f satisfies the inequality

$$-\langle Q(f), \ln f\rangle + H(T) + \left\langle \chi^\pm f, \left|\frac{f}{(f,1)_\pm}\right|\right\rangle_\pm \leqslant C(T) \tag{4.9}$$

with $C(T)>0$ depending only on f_0 and on K_2, K_3, K_4.

Proof. Due to (K_1), whenever H^- and H^+ are defined,

$$H^- - H^+ = \left\langle \chi^- f, \ln \frac{f}{(f,1)_-} \right\rangle_- - \left\langle \chi^+ f, \ln \frac{f}{(f,1)_+} \right\rangle_+ \tag{4.10}$$

Hence using Green's formula (2.2) and approximation, we have

$$-\langle Q, \ln f\rangle + H(T) + \left\langle \chi^- f, \ln \frac{f}{(f,1)_-} \right\rangle_- \leqslant H(0) + \left\langle \chi^+ f, \frac{\ln f}{(f,1)_+} \right\rangle_+ \tag{4.11}$$

Since mass and energy are bounded by (4.1)–(4.2), Carleman's lemma[3] on lower bounds for H can be applied to give

$$-H(T) \leqslant C(T)$$

with $C(T)$ depending on f_0, and $C_1(T)$ of Lemma 4.1. Also (4.10)–(4.11) together with (K_4) and Lemma 4.1 imply

$$-\langle Q, \ln f\rangle + \left\langle \chi^- f, \ln \frac{f}{(f,1)_-} \right\rangle_- \leqslant C(T)$$

For any measure μ on R^3 and any $g > 0$ with $(1 + |\xi|^2)(g + e^{-\xi^2}) \in L^1_\mu(\mathbb{R}^3)$, it holds that

$$-\int_{\mathbb{R}^3} g \ln g 1_{g<1}\, d\mu \leqslant C + \int_{\mathbb{R}^3} (1 + |\xi|^2)\, g\, d\mu$$

In particular, $g = f/(f, 1)_\pm$, $d\mu = \chi^\pm\, |\xi \cdot n|\, d\xi$ gives

$$\left\langle \chi^\pm f, \ln \frac{f}{(f,1)_\pm} \right\rangle_\pm \geqslant \left\langle \chi^\pm f, \left|\ln \frac{f}{(f,1)_\pm}\right| \right\rangle_\pm - C(1 + q_0^\pm + q_2^\pm)$$

This together with (4.10)–(4.11) and (K_4) gives

$$-\langle Q, \ln f\rangle + H(T) + \left\langle \chi^\pm f, \left|\ln \frac{f}{(f,1)_\pm}\right| \right\rangle_\pm \leqslant C(T) \quad \blacksquare$$

Lemma 4.3. Assume that (K_0), (K_1), and (K_5) hold, and that

$$0 \leqslant f_0 \in L^1(\Omega \times \mathbb{R}^3), \qquad Q^+ \in L^1(\mathscr{D})$$

Then any solution of (2.20)–(2.21), (1.2)–(1.3) in $L^1(\mathscr{D})$ with $\nu f \in L^1(\mathscr{D})$ satisfies

$$\varphi(T - t)\, \gamma^\pm f \in L^1(\Gamma^\pm)$$

with $\varphi(s) = s\psi(s)$ and ψ defined by (K_5).

Proof. By hypothesis $f, \wedge f \in L^1(\mathscr{D})$. Hence by Lemma 3.1

$$s^- f^+ \in L^{1+}$$

By (K_5)

$$\begin{aligned}\|s^-\gamma^+ f\|_{L^1(\Gamma^+)} &= \langle \chi^+ s^-, K\gamma^- f\rangle_+ \\ &= \langle \chi^-\gamma^- f, K^* s^-\rangle_- \geqslant \langle \chi^-\varphi(T-t), \gamma^- f\rangle_-\end{aligned}$$

Finally,

$$\langle \chi^-\varphi(T-t), \gamma^- f\rangle_- = \langle \chi^+\varphi(T-t), \gamma^+ f\rangle_+ \quad \blacksquare$$

5. PROBLEMS WITH GIVEN Q^+ AND ν

In this section we discuss problems with Q^+ and ν given. The principal results are contained in the following two theorems.

Theorem 5.1. Assume that (K_0)–(K_3) hold. Consider the problem (1.1)–(1.3) with f_0, Q^+, and ν given and positive together with

$$(1+|\xi|^2) f_0 \in L^1(\Omega\times\mathbb{R}^3), \qquad (1+|\xi|^2) Q^+ \in L^1(\mathscr{D}), \qquad \nu \in L^1_{\text{loc}}(\mathscr{D})$$

It has a unique, nonnegative, exponential solution with $\mu_s^\pm = 0$ and equality in (2.19). Moreover, $Q \in L^1(\mathscr{D})$, and

$$(1+|\xi|^2)\gamma^\pm f \in L^{1\pm} \tag{5.1}$$

$$(1+|\xi|^2) f \in C([0,T], L^1(\Omega\times\mathbb{R}^3)) \tag{5.2}$$

Theorem 5.2. Assume that (K_0), (K_1), and (K_5) hold. Consider the problem (1.1)–(1.3) with f_0, Q^+, and ν given and positive, and

$$f_0 \in L^1(\Omega\times\mathbb{R}^3), \qquad Q^+ \in L^1(\mathscr{D}), \qquad \nu \in L^1_{\text{loc}}(\mathscr{D})$$

It has a unique, nonnegative, exponential solution with equality in (2.19) and

$$\mu_s^\pm = 0, \qquad f \in C([0,T], L^1(\Omega\times\mathbb{R}^3))$$

In addition,

$$\langle f, 1\rangle_t = \langle f, 1\rangle_0 \quad \text{if} \quad (Q, 1) = 0 \tag{5.3}$$

Proof of Theorem 5.2. An existence result is proved in ref. 1 for general kernels only satisfying (K_0), (K_1). In that case

$$\inf_{t\leqslant T} \langle f, 1\rangle_t \leqslant \langle f_0, 1\rangle_0$$

holds when $(Q, 1)=0$, instead of (5.3) in our case. The constructions in the existence proof below are used in the following uniqueness proof.

Let $\lambda \in (0, 1)$. By Lemma 2.1 the operator $\lambda \mathscr{K}^+$, where

$$\mathscr{K}^+ f^+ = \mathscr{K}(0, f^+, 0, \nu) \tag{5.4}$$

is a contraction in L^{1+}. Hence the equation

$$f^+ = \lambda \mathscr{K}^+ f^+ + \mathscr{K}(f_0, 0, Q^+, \nu)$$

has a unique solution in L^{1+}. Denote by $\mathscr{R}_\lambda$ the solution operator

$$\mathscr{R}_\lambda = (I - \lambda \mathscr{K}^+)^{-1}$$

Clearly,

$$f^\lambda = V(f_0, \mathscr{R}_\lambda \mathscr{K}(f_0, 0, Q^+, \nu), Q^+, \nu) \tag{5.5}$$

is an exponential solution of (1.1)–(1.3) where K is replaced by λK. Moreover, $\mu_s^{\pm\lambda} = 0$ for the solution f^λ and (2.19) holds with equality. Using the equivalence between exponential and weak solutions,[4] we can apply Green's formula (2.2). With $g=1$ this gives

$$\|f^\lambda(T)\|_{L^1(\Omega \times \mathbb{R}^3)} + \|\nu f^\lambda\|_{L^1(\mathscr{D})} \leqslant \|f_0\|_{L^1(\Omega \times \mathbb{R}^3)} + \|Q^+\|_{L^1(\mathscr{D})} \tag{5.6}$$

Taking a sequence $\lambda \uparrow 1$, we obtain a nondecreasing sequence f^λ. Hence by (5.6) the limit f in $L^1(\mathscr{D})$ is a limit in L^1 and a.e. sense.

By Lemma 3.1 and (5.6)

$$\langle s^\pm, \gamma^\mp f \rangle_\mp \leqslant CT(\|Q^+\|_{L^1(\mathscr{D})} + \|f_0\|_{L^1(\Omega \times \mathbb{R}^3)})$$

f is a solution of (2.20)–(2.21), (1.2)–(1.3) with $Q \in L^1(\mathscr{D})$. Lemma 4.3 implies that

$$\varphi(T-t)\, \gamma^\pm f \in L^1(\Gamma^\pm) \tag{5.7}$$

Hence f is an exponential solution. For $(Q, 1)=0$, (5.3) follows from (5.7) and (2.2). Integration of the equivalent mild form gives

$$\begin{aligned} &\|f^\#(s) - f^\#(\tau)\|_{L^1(\Omega \times \mathbb{R}^3)} \\ &\quad \leqslant \int_s^\tau \langle |Q^\#(z)|, 1\rangle_z \, dz + \int_s^\tau dz + \int_s^\tau dz \int_{\partial\Omega} d\sigma \int_{\mathbb{R}^3} \gamma f \, |\xi \cdot n| \, d\xi \end{aligned} \tag{5.8}$$

Hence $f \in C([0, T], L^1(\Omega \times \mathbb{R}^3))$.

One way of obtaining an exponential solution with $\mu_s^\pm = 0$ and equality in (2.19) is the following. Define

$$f^{+n} = \sum_{j=0}^{n} (\mathcal{K}^+)^j \mathcal{K}(f_0, 0, Q^+, \nu) \tag{5.9}$$

$$f^n = V(f_0, f^{+n}, Q^+, \nu) \tag{5.10}$$

Clearly,

$$0 \leqslant f^{n-1} \leqslant f^n \leqslant f \tag{5.11}$$

where f is the solution described above. Hence there exists $\underline{f} \in L^1(\mathcal{D})$, such that $f_n \to \underline{f}$ in $L^1(\mathcal{D})$ and a.e. Evidently $\underline{f}$ is an exponential solution, and for any nonnegative exponential solution F with $\mu_s^\pm = 0$ and equality in (2.19),

$$\underline{f} \leqslant F \tag{5.12}$$

In this sense $\underline{f}$ is the minimal solution.

We next prove that the solution is unique in the class of exponential solutions with equality in (2.19) and $\mu_s^\pm = 0$. Green's formula (2.2) can be applied to any such solution F giving

$$\langle \nu, F - \underline{f} \rangle + \langle F - \underline{f}, 1 \rangle_t = 0 \tag{5.13}$$

And so uniqueness follows by (5.12). ▮

Proof of Theorem 5.1. The existence of a solution of (2.20)–(2.21), (1.2)–(1.3) follows as in the proof of Theorem 5.2. It follows from (5.6) that $Qf \in L^1(\mathcal{D})$. From Lemma 4.1, applied to the exponential approximations f^λ, and arguing as in (5.6) with $g = 1 + \xi^2$, it follows that the solution f is of exponential type and satisfies the conclusions of Lemma 4.1. In particular $(1 + \xi^3) Q(f) \in L^1(\mathcal{D})$ and (5.1) holds. From here (5.2) can be deduced similarly to (5.8) of Theorem 5.2.

The uniqueness among solutions with $\mu_s^- = 0$ and equality in (2.19) are consequences of (2.2) applied to $F - \underline{f}$. This again gives (5.13), thus uniqueness using (5.12). ▮

Corollary 5.3. Let f be an exponential solution of (1.1)–(1.3) with $\mu_s^- = 0$ and equality in (2.19). Then

$$\gamma^+ f \geqslant \sum_{j=0}^{n} (\mathcal{K}^+)^j \mathcal{K}(f_0, 0, Q^+, \nu)$$

This follows directly from (5.9).

Corollary 5.4. Keep the assumptions of Theorem 5.1. Assume in addition that $Q^{\pm}$ are Lipschitz operators in

$$L^1(\Omega \times \mathbb{R}^3, \varphi\, dx\, d\xi) \qquad \text{where} \quad \varphi = (1 + |\xi|^2)$$

with $\nu(f) \in L^1_{\text{loc}}(\mathscr{D})$ for $f \in L^1(\mathscr{D}, \varphi\, dt\, dx\, d\xi)$, and with

$$\langle Qf, 1 \rangle = \langle Qf, \xi^2 \rangle = 0 \tag{5.14}$$

Then the problem (1.1)–(1.3) has a unique exponential solution with $\mu_s^{\pm} = 0$ and equality in (2.19).

Proof. Fix f_0 and denote the solution of Theorem 5.1 by $\mathscr{R}(Q)$. The continuity properties of $\mathscr{R}$ implied by the proof of Lemma 4.1 prove that $\mathscr{R} \circ Q$ is locally Lipschitz in $C([0, T], L^1(\Omega \times \mathbb{R}^3, \varphi\, dx\, d\xi))$. Existence and uniqueness for small T can be established by a contraction fixed-point argument. Due to (5.13) and Lemma 4.1, the solution can be extended to an arbitrary time interval.

Remark. All results of this section are valid for the boundary condition

$$\gamma^+ f = K\gamma^- f + g \qquad \text{on} \quad \Gamma^+$$

with g prescribed such that $(1 + |\xi|^2)\, g \in L^{1+}$.

6. THE BOLTZMANN EQUATION

In this section we consider the Boltzmann equation for the full class of collision operators studied in ref. 4 and with $\partial\Omega$ of Lyapunov type. The results of ref. 1 are generalized to the case of varying boundary temperature and unbounded velocities. For that extension some energy control at the boundary seems necessary. Here we assume (K_0)–(K_4).

Theorem 6.1. Assume that

$$(1 + |\xi|^2)\, f_0,\, f_0 \log f_0 \in L^1(\Omega \times \mathbb{R}^3), \qquad f_0 \geqslant 0$$

Then there exists an exponential solution (Definition 2.2) of (1.1)–(1.3) satisfying

$$f \in C([0, T], L^1(\Omega \times \mathbb{R}^3), \qquad f \geqslant 0, \qquad \langle f, 1 \rangle_t = \langle f_0, 1 \rangle_0$$

$$(1 + |\xi|^2)\, \gamma^{\pm} f \in L^{1\pm} \tag{6.1}$$

$$\sup_{t \geqslant T} [\langle f, \ln f \rangle_t + \langle f, |\xi|^2 \rangle_t] + \langle \gamma^{\pm} f, 1 + |\xi|^2 \rangle_{\pm} + \langle e(f), 1 \rangle \leqslant C(T) \tag{6.2}$$

Here

$$e(f)(\mathrm{t}) = \frac{1}{4} \int_{R^3} \int_{B^+} (f'f'_* - ff_*) \log \frac{f'f'_*}{ff_*} B(|\xi - \xi_*|, u) \, d\xi_* \, du$$

with B^+ the domain of integration of the angular variable u.

Proof. Following the approximation scheme of ref. 4, we replace the collision kernel B by

$$B_n = [1 + n^{-1}(\cdot, 1)]^{-1} \chi(n - |\xi|^2 - |\xi_*|^2)(B \wedge n)$$

where χ is the characteristic function of R_+. Let Q_n, Q_n^+, ν_n be the corresponding operators with B replaced by B_n. Then with $\varphi(\xi) = 1 + |\xi|^2$,

$$\|\varphi(Q_n f - Q_n g)\|_{L^1(\Omega \times \mathbb{R}^3)} \leqslant C(n) \, \|\varphi(f - g)\|_{L^1(\Omega \times \mathbb{R}^3)}$$

By Corollary 5.4 for every n the problem (1.1)–(1.3) with $Qf = Q_n f$ has a unique exponential solution f^n with $\mu_s^{\pm n} = 0$, equality in (2.19), and mass conservation. Set

$$\mathscr{E} = \{(g, g^\pm); \, g = f^n, \, g^\pm = \chi^\pm \gamma^\pm f^n\}$$

Lemmas 4.1 and 4.2 imply that

$$\sup_{g \in \mathscr{E}} \{\langle g, 1 + |\xi|^2 \rangle_t + \langle g, \ln g \rangle_t + \langle e(g), 1 \rangle + \langle \gamma^- g, 1 + |\xi|^2 \rangle_- + \langle \gamma^+ g, 1 + |\xi|^2 \rangle_+ \} \leqslant C(T) \tag{6.3}$$

So $\mathscr{E}$ is weak-weak* compact in $L^1(\mathscr{D}) \times \mathscr{M}^\pm$. Thus we may assume that $(f^n, \chi^\pm \gamma^\pm f^n)$ converges in weak-weak* sense in $L^1(\mathscr{D}) \times \mathscr{M}^\pm$ to some $(f, \mu^\pm)$ (after choosing a suitable subsequence).

Evidently[4] f satisfies the Boltzmann equation in the interior of $\mathscr{D}$ and with initial value f_0. It follows from (6.3) that (6.2) holds, i.e.,

$$\langle f, 1 + |\xi|^2 \rangle_t + \langle f, \ln f \rangle_t + \langle e(f), 1 \rangle \leqslant C(T)$$

And so $f(t, \cdot)$ conserves mass and is continuous from $\mathbb{R}_+$ to $L^1(\Omega \times \mathbb{R}^3)$. Also $\mu^\pm$, the weak* limits in $\mathscr{M}^\pm$ of $\chi^\pm \gamma^\pm f^n$, satisfy (2.18), since $\chi^\pm \gamma^\pm f^n$ satisfy (1.2) and since by (6.3)

$$\sup_n \langle \gamma^\pm f^n, 1 + |\xi|^2 \rangle_\pm \leqslant C(T) \tag{6.4}$$

The nonintegrability of $Q^\pm$ may influence the trace properties, and is the reason we only obtain (2.19). The proof of (2.19) is carried out for μ_c^+ and

$\chi^+\gamma^+f$. The case of μ_c^- and $\chi^-\gamma^-f$ is analogous. Evidently (6.1) follows from (2.19) and (6.4).

Take $\mathrm{r}_0=(t, x, \xi)\in\Gamma^+$, $0<t<T$. Since $\partial\Omega$ is of Lyapunov type, there is an open neighborhood $\mathcal{N}$ of r_0 in Γ^+ and $\varepsilon>0$ such that $R^s(\bar{\mathcal{N}})\subset\mathcal{D}$ for $0<s\leqslant\varepsilon$. It is enough to prove that

$$f_c\,d\sigma_\mathrm{r}^\pm=d\mu_c^\pm\geqslant\gamma^\pm f\,d\sigma_\mathrm{r}^\pm \tag{6.5}$$

in $\mathcal{N}$ outside of some subset, provided the Lebesgue measure of that subset can be made arbitrarily small.

For a.e. $\mathrm{r}\in\mathcal{N}$ it holds for all $n\in\mathbb{N}$ and all $0<\delta\leqslant\varepsilon$ that

$$|f_0^nR(s,\mathrm{r})-f^n(\mathrm{r})|\leqslant\max\left(\int_0^\delta Q_n^{+\,\#}(s,\mathrm{r})\,ds,\int_0^\delta Q_n^{-\,\#}(s,\mathrm{r})\,ds\right)$$

Now for $j>1$

$$f'f'_*\leqslant jff_*+\frac{1}{\log j}(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}$$

and by (6.3) for all $n\in\mathbb{N}$

$$\langle e(f)^n,1\rangle\leqslant C(T)$$

Let ψ be the characteristic function of a measurable subset of $\mathcal{N}$. To prove (6.5), it suffices to prove

$$\lim_{\delta\to0}\sup_{n\in\mathbb{N}}\int_\mathcal{N}|f^{n\#}(\delta,\mathrm{r})-f^n(\mathrm{r})|\,\psi\,d\sigma_\mathrm{r}^+=0$$

for ψ's which vanish only on arbitrarily small subsets of $\mathcal{N}$. But this follows if

$$\lim_{\delta\to0}\sup_{n\in\mathbb{N}}\int_\mathcal{N}\psi\,d\sigma_\mathrm{r}^+\int_0^\delta f^{n\#}(s,\mathrm{r})\,v^\#(f^n)(s,\mathrm{r})\,ds=0$$

for such ψ's.

For a.e. $0<s<\varepsilon$, by averaging

$$\int(R^+\psi)^\#(s)\,f^{n\#}(s)\,d\xi \tag{6.6}$$

converges strongly in L^1 to

$$\int(R^+\psi)^\#(s)\,f^\#(s)\,d\xi$$

Fix $s = s_0$ with $0 < s_0 < \varepsilon$ so that this holds. Also outside of an arbitrarily small subset of $\mathcal{N}$, by averaging,

$$\int_0^s \nu^\#(f^n)(\tau, \mathbf{r})\, d\tau$$

converges uniformly (in $\mathbf{r}$) to

$$\int_0^s \nu^\#(f)(\tau, \mathbf{r})\, d\tau$$

and the latter is uniformly bounded by

$$\int_0^\varepsilon \nu^\#(f)(\tau, \mathbf{r})\, d\tau \leqslant C_0 < \infty$$

Let ψ be the characteristic function of such a (big) subset of $\mathcal{N}$. For $0 < s < s_0$ and n large

$$\psi f^{n\#}(s, \mathbf{r}) \leqslant \psi f^{n\#}(s_0, \mathbf{r})\, \Pi^+(\nu_n, 0, s_0, \mathbf{r}) \leqslant C_1 \psi f^{n\#}(s_0, \mathbf{r}) \tag{6.7}$$

By the strong L^1 convergence of (6.6), after removing an arbitrarily small set of (t, x) such that $(t, x, \xi) \in \mathcal{N}$ for some ξ, and letting ψ' be the characteristic function of the rest of $\mathcal{N}$, we have

$$\int \psi\psi' f^{n\#}(s_0, \mathbf{r})\, d\xi \leqslant C_2 < \infty, \qquad n \in \mathbb{N} \tag{6.8}$$

Also outside of an arbitrarily small subset of $\mathcal{N}$,

$$\int_0^\delta \nu^\#(f)(s, \mathbf{r})\, ds \tag{6.9}$$

can be made arbitrarily small by picking δ small enough. And so using (6.7), we have

$$\int_{\mathcal{N}} \psi\psi'\, d\sigma_{\mathbf{r}}^+ \int_0^\delta f^{n\#}(s, \mathbf{r})\, \nu^\#(f^n)(s, \mathbf{r})\, ds$$
$$\leqslant \int_{\mathcal{N}} \psi\psi'\, d\sigma_{\mathbf{r}}^+\, f^{n\#}(s_0, \mathbf{r}) \int_0^\delta \nu^\#(f^n)(s, \mathbf{r})\, ds$$

where by (6.8)–(6.9) the right-hand side can be made arbitrarily small for $n > n_0$ by choosing δ small enough and then n_0 large enough. This completes the proof of the theorem. ∎

7. THE BGK EQUATION

Consider the problem (1.1)–(1.3) with Maxwellian diffuse reflection and

$$Qf = \mathscr{M}f - f \tag{7.1}$$

where

$$\mathscr{M}f = \rho(2\pi)^{-3/2}\Theta^{3/2}\exp\left(-\frac{\Theta}{2}|\xi - u|^2\right)$$

$$\rho = (f, 1), \qquad \rho u = (f, \xi), \qquad \rho(|u|^2 + \Theta^{-1}) = (f, |\xi|^2)$$

Set

$$\varphi_\beta(\xi) = (1 + |\xi|)^\beta$$

$$X = X_{\infty,\beta} = \{f: \mathscr{D} \to \mathbb{R};\ \varphi_\beta f \in L^\infty(\mathscr{D})\}$$

$$X^\pm = X^\pm_{\infty,\beta} = \{f: \Gamma^\pm \to \mathbb{R};\ \varphi_\beta f \in L^\infty(\Gamma^\pm)\}$$

$$X^0 = X^0_{\infty,\beta} = \{f: \Gamma^s \to \mathbb{R};\ \varphi_\beta f \in L^\infty(\Gamma^s)\}$$

The natural norms in X, $X^\pm$, X^0 are

$$\|f\|_X = \|f\|_{L^\infty(\mathscr{D})}$$

$$\|f\|_{X^j} = \|f\|_{L^\infty(\Gamma^j)}, \qquad j = +, -, 0$$

The aim of this section is to prove that under suitable conditions, the problem (1.1)–(1.3) in the present setting has a unique solution in $X_{\infty,\beta}$ with $\beta > 5$.

Let $\partial\Omega$ be a Lyapunov surface. Assume that the initial value of f_0 satisfies

$$0 \leqslant f_0 \in X^0_{\infty,\beta} \qquad \text{for some} \quad \beta > 5 \tag{7.2}$$

$$\inf_{t\leqslant T}\inf_{x\in\Omega}\sup_n\left(\sum_{j=0}^n V^+((\mathscr{K}^+)^j\mathscr{K}(f_0, 0, 0, 1) + V(f_0, 0, 0, 1\right) = \underline{\rho} > 0 \tag{7.3}$$

Here $V^+(\cdot) = V(0, \cdot, 0, 1)$, and the operators $\mathscr{K}^+$, $\mathscr{K}$, and V are defined by (5.4), (2.6), and (2.4). Condition (7.3) provides a nonzero lower bound for the density $\rho = (f, 1)$ of a solution of the problem (1.1)–(1.3) with $Q^+ = 0$, $\nu = 1$.

Theorem 7.1. Let Q be given by (7.1). Assume that f_0 satisfies (7.2) and (7.3) with $\|f_0\|_{X^0_{\infty,\beta}} \leqslant C_0$. Then there exists a unique exponential solution f of (1.1)–(1.3) in $X_{\infty,\beta}$ with ($\mu_s^\pm = 0$, equality in (2.19), and)

$$\inf_{t\leqslant T}\inf_{x\in\Omega}(f, 1) \geqslant \underline{\rho}$$

In addition,

$$\|f\|_X + \|\gamma^- f\|_{X^-} + \|\gamma^+ f\|_{X^+} \leqslant C(T) \tag{7.4}$$

Here $C(T)$ depends only on T, $\underline{\rho}$, β, and C_0 when

$$\|f_0\|_{X^0_{\infty,\beta}} \leqslant C_0$$

Two main tools in the proof are the following consequences of ref. 8:

(i) For every $\beta > 5$, there is $C(\beta) > 0$ such that

$$\|\varphi_\beta \mathcal{M} f\|_{L^\infty(\mathbb{R}^3)} \leqslant C(\beta) \, \|\varphi_\beta f\|_{L^\infty(\mathbb{R}^3)} \tag{7.5}$$

(ii) With

$$\mathscr{A} = \{f;\, \|f\|_{X^0_{\infty,\beta}} \leqslant C, \inf(f, 1) \geqslant \underline{\rho}, f \geqslant 0\} \tag{7.6}$$

there is a constant $C(\mathscr{A})$ depending on $\mathscr{A}$ such that

$$\|\varphi_2(\mathcal{M}f - \mathcal{M}g)\|_{L^1(\Omega \times \mathbb{R}^3)} \leqslant C(\mathscr{A}) \, \|\varphi_2(f-g)\|_{L^1(\Omega \times \mathbb{R}^3)} \qquad \text{for} \quad f, g \in \mathscr{A} \tag{7.7}$$

In the proof of Theorem 7.1 the following result is needed.

Proposition 7.2. Assume that $\nu = 1$, $f_0 \in X^0_{\infty,\beta}$, $Q^+ \in X_{\infty,\beta}$ are positive with $\beta > 5$. Then for $T > 0$, there exists a unique exponential solution of (1.1)–(1.3) with $\mu_s^\pm = 0$, equality in (2.19), and belonging to $X_{\infty,\beta}$. In addition,

$$\|f\|_X + \|\gamma^- f\|_{X^-} + \|\gamma^+ f\|_{X^+} \leqslant C(T)(\|f_0\|_{X^0} + T \, \|Q^+\|_X) \tag{7.8}$$

with $C(T)$ independent of f_0 and Q^+, and

$$\inf_{t \leqslant T} \inf_{x \in \Omega} (f, 1) \geqslant \underline{\rho} \tag{7.9}$$

where $\underline{\rho}$ is defined by (7.3).

By (1.4)–(1.5)

$$f^+(\mathfrak{r}) = M(\mathfrak{r}) \, q(t, x) \tag{7.10}$$

where

$$q(t, x) = (f(t, x, \cdot), 1)_+, \qquad \mathfrak{r} = (t, x, \xi)$$

It follows from (2.5) that if f is the solution of Proposition 7.2, then q solves the following equation:

$$q = \mathscr{L} q + F(f_0, Q^+) \tag{7.11}$$

Here

$$\mathscr{L}q = (\mathscr{K}(0, Mq, 0, 1), 1)_+$$
$$F(f_0, Q^+) = (\mathscr{K}(f_0, 0, Q^+, 1), 1)_+$$

For our proof of Proposition 7.2, a preliminary study of (7.11) is needed.

In explicit form $\mathscr{L}$ and F are given by

$$\mathscr{L}q(t, x) = \int_{\xi \cdot n(x) < 0} q^{\#}(-t^+, \mathrm{r})\, M^{\#}(-t^+, \mathrm{r})\, 1_{t^+(\mathrm{r}) < t} \exp(-t^+(\mathrm{r}))\, |\xi \cdot n(x)|\, d\xi$$

$$F(t_0, Q^+) = \int_{\xi \cdot u(x) < 0} \left[f_0(x - t\xi, \xi)\, 1_{t^+(\mathrm{r}) > t} \exp(-t) + \int_{-s^+(\mathrm{r})}^{0} Q^{+\#}(\tau, \mathrm{r}) \exp(-\tau)\, d\tau \right] |\xi \cdot n(x)|\, d\xi \tag{7.12}$$

Set $s = t - t^+(\mathrm{r})$, $y = x - t^+(\mathrm{r})\xi$ for $s \in [0, T]$, $y \in \partial\Omega$. The change of variables $\xi \to (s, y)$ leads to the following representation of $\mathscr{L}$:

$$\mathscr{L}q(t, x) = \int_0^t e^{s-t}\, ds \left[\int_{\partial\Omega} \mathscr{L}(t, x, s, y)\, q(s, y)\, d\sigma_y \right]$$

where the kernel $\mathscr{L}$ is defined by

$$\mathscr{L}(t, x, s, y) = |(x - y) \cdot n(x)| \cdot |(x - y) \cdot n(y)| \cdot |t - s|^{-d-2} M(s, y, (x - y)(t - s)^{-1}) \tag{7.13}$$

Since $\partial\Omega$ is Lyapunov, it follows that

$$\delta(\varepsilon) := \inf_{\mathcal{O}_\varepsilon} |y - x| > 0 \qquad \text{with} \quad \mathcal{O}_\varepsilon = \{(x, \xi);\ -\varepsilon\, |\xi| > \xi \cdot n(x)\}$$

By hypothesis

$$0 \leqslant M(t, x, \xi) \leqslant (2\pi)^{-2} c_2^2 \exp\left(-\frac{c_1}{2} |\xi|^2 \right) \tag{7.14}$$

So in $\mathcal{O}_\varepsilon$

$$\mathscr{L}(t, x, s, y) \leqslant \delta^{-d} \left| \frac{x - y}{t - s} \right|^{d+2} M\left(s, y, \frac{x - y}{t - s}\right) \leqslant C\delta^{-d} \tag{7.15}$$

Next, Eq. (7.11) will be solved.

Lemma 7.3. Assume that f_0, Q^+ are positive, that $(1+|\xi|^2)\,Q^+ \in L^1(\mathscr{D})$, and that $F(f_0, Q^+) \in L^\infty((0, T)\times\partial\Omega)$. Then Eq. (7.11) has a unique solution in $L^\infty((0, T)\times\partial\Omega)$.

Proof. It is sufficient to consider the case of f_0, $Q^+ \geqslant 0$. Theorem 5.1 implies that (7.11) has a solution q in $L^1((0, T)\times\partial\Omega)$. Any solution of (7.11) generates [cf. (2.7)–(2.8)] an exponential solution of (1.1)–(1.3) with $\mu_s^\pm = 0$ and equality in (2.19), and so the uniqueness in L^1 follows from Theorem 5.1. Since $L^\infty((0, T)\times\partial\Omega) \subset L^1((0, T)\times\partial\Omega)$, the uniqueness also holds in L^∞.

By (7.14)

$$\sup_{t\in[0,T]}\ \sup_{x\in\partial\Omega}\int_{-\varepsilon|\xi|<\xi\cdot n(x)<0} M^{\#}(-t^+, \mathrm{r})\,|\xi\cdot n(x)|\,d\xi$$

$$\leqslant C\int_{-\varepsilon|\xi|=\xi\cdot n(x)<0}\exp\left(-\frac{c_1}{4}|\xi|^2\right)d\xi<\frac{1}{2}$$

for some $\varepsilon_0>0$ and $0<\varepsilon\leqslant\varepsilon_0$.

Hence

$$|\mathscr{L}q(t,x)|\leqslant\tfrac{1}{2}\,\|q\|_\infty+C\delta^{-d}\int_0^t ds\int_{\partial\Omega}q(s,y)\,d\sigma_y$$

and so

$$\|\mathscr{L}q\|_\infty\leqslant\tfrac{1}{2}\,\|q\|_\infty+c\,\|q\|_{L^1((0,T)\times\partial\Omega)}$$

Using this and the properties of $F(f_0, Q^+)$ together with (7.11), we conclude that $q\in L^\infty$. ∎

Proof of Proposition 7.2. By (7.12) and Hölder's inequality

$$\|F(f_0, Q^+)\|_\infty\leqslant C(T)(\|f_0\|_{X^0}+T\,\|Q^+\|_X) \tag{7.16}$$

with $C(T)$ independent of f_0 and Q^+.

By Theorem 5.1 the problem (1.1)–(1.3) has in the present case a unique exponential solution in L^1 with $\mu_s^\pm=0$ and equality in (2.19).

The corresponding q satisfies (7.11). So by Lemma 7.3, q is in L^∞ and by its proof,

$$\|q\|_\infty\leqslant C(\|q\|_1+\|F(f_0, Q^+)\|_\infty) \tag{7.17}$$

with C independent of f_0 and Q^+. Just as in the proof of Lemma 4.1, we have

$$\|q\|_1\leqslant C(T)(\|f_0\|_1+T\,\|Q^+\|_1)\leqslant C(T)(\|f_0\|_{X^0}+T\,\|Q^+\|_X) \tag{7.18}$$

with $C(T)$ independent of f_0 and Q^+. By (7.16)–(7.18)

$$\|q\|_\infty \leqslant C(T)(\|f_0\|_{X^0} + T\,\|Q^+\|_X)$$

with $C(T)$ independent of f_0 and Q^+. By (7.10) this implies

$$\|f^+\|_{X^+} \leqslant C(T)(\|f_0\|_{X^0} + T\,\|Q^+\|_X) \tag{7.19}$$

It follows from (2.4) that

$$\|\gamma^- f\|_{X^-} + \|f\|_X \leqslant C(T)(\|f_0\|_{X^0} + \|f^+\|_{X^+} + T\,\|Q^+\|_X) \tag{7.20}$$

Together (7.19)–(7.20) imply (7.8).

Using (5.9)–(5.11) and (7.3), we find that the bound (7.9) follows. ∎

Proof of Theorem 7.1. Given f_0 satisfying (7.2)–(7.3), fix $\underline{\rho}$ as defined by (7.3). Let $f = \mathscr{R}(Q^+)$ denote the solution of (1.1)–(1.3) according to Proposition 7.2 for given $Q^+ \in X_{\infty,\beta}$. By (7.9)

$$\inf_{t \leqslant \tau}\ \inf_{x \in \Omega} (f, 1) \geqslant \underline{\rho}$$

and by (7.8)

$$\|f\|_{X_{\infty,\beta}} \leqslant C(T)(\|f_0\|_{X^0_{\infty,\beta}} + T\,\|Q^+\|_{X_{\infty,\beta}}) \tag{7.21}$$

with $C(T)$ independent of f_0 and Q^+.

Consider for $T \leqslant 1$ the iteration $f^0 = f_0$, $f^n = \mathscr{R} \circ \mathscr{M} f^{n-1}$ for $n \geqslant 1$. By (7.5)

$$\|\mathscr{M} f^{n-1}\|_{X^0_{\infty,\beta}} \leqslant C(\beta)(\|f^{n-1}\|_{X^0_{\infty,\beta}})$$

and so by (7.8)

$$\|f^n\|_{X_{\infty,\beta}} \leqslant C(T)(\|f_0\|_{X^0_{\infty,\beta}} + TC(\beta)\,\|f^{n-1}\|_{X_{\infty,\beta}}) \tag{7.22}$$

In the definition of $\mathscr{A}$ in (7.6) choose

$$C = 2 \sup_{T \leqslant 1} (1 + C(T))\,\|f_0\|_{X^0_{\infty,\beta}}$$

Take $T \leqslant 1$ so small that

$$TC(\beta) \sup_{T \leqslant 1} (1 + C(T)) \leqslant \tfrac{1}{2}$$

By induction it follows from (7.22) that for all n

$$\|f^n\|_{X_{\infty,\beta}} \leqslant C(T)(\|f_0\|_{X^0_{\infty,\beta}} + TC(\beta)\,\|f^{n-1}\|_{X_{\infty,\beta}}) \leqslant C \tag{7.23}$$

i.e., $f^n \in \mathscr{A}$ for all n.

By (7.6)–(7.7) the operator $\mathscr{M}$ is a Lipschitz operator on $\mathscr{A}$ with respect to the norm in $L^1(\Omega \times R^3,\ \varphi_2\, dx\, d\xi)$. So by Corollary 5.4 the problem (1.1)–(1.3) has a unique m-exponential solution in the present case. Moreover, $f = \lim f^n$ in L^1 sense, and so by (7.23), $f \in \mathscr{A}$, hence $f \in X_{\infty,\beta}$.

By (7.5), $\mathscr{M}f \in X_{\infty,\beta}$, and so f is the type of solution in $X_{\infty,\beta}$ considered in Proposition 7.2. Hence (7.4) is a consequence of (7.8) and (7.5). This concludes the proof of the theorem for the chosen T. By induction the theorem holds for all $T \leqslant 1$, and then for all $T < \infty$. ∎

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